MA 3046 - Matrix Analysis
Review of Fundamental Matrix Spaces

(1) The essential, defining properties of (sub)spaces are that they are nonempty and "closed" under *addition* and *scalar multiplication.*

(2) The only nontrivial subspaces in $\mathbb{R}^2$ are lines *through the origin.* The only ones in $\mathbb{R}^3$ are lines and planes *through the origin.*

(3) The fundamental question, given a (sub)space, is what is the *minimum information* necessary in order to be able to completely (re)construct that space using only addition and scalar multiplication.

(4) *Span* (of a given set of vectors) is the term used to describe all of the possible vectors that can be (re)constructed (from the vectors in that set) using only addition and scalar multiplication.

(5) *Linear Independence* (of a given set of vectors) is the term used to describe whether any vectors in that set are *redundant* in the sense that one or more can be discarded without changing the span (of that set).

(6) *Basis* is the term used to describe a linearly independent spanning set. A basis represents the minimum information necessary to (re)constructed a (sub)space (from the vectors in that set) using only addition and scalar multiplication.

(7) Bases are not unique, but the *number of vectors* in a basis for any given (sub)space is! This number is called the *dimension* of that (sub)space.

(8) The four fundamental subspaces associated with a given real matrix ($\mathbf{A} \in \mathbb{R}^{m\times n}$) are:
   (i) The *Column Space* of $\mathbf{A}$: $\left\{\mathbf{b} \middle| \mathbf{A}\,\mathbf{x} = \mathbf{b} \text{ is solvable}\right\} \equiv$ Span of the columns of $\mathbf{A}$
   (ii) The *Null Space* of $\mathbf{A}$: $\left\{\mathbf{x} \middle| \mathbf{A}\,\mathbf{x} = 0\right\}$
   (iii) The *Null Space of* $\mathbf{A}^T$: $\left\{\mathbf{y} \middle| \mathbf{A}^T\,\mathbf{y} = \mathbf{0}\right\} \equiv \left\{\mathbf{y} \middle| \mathbf{y}^T\,\mathbf{A} = \mathbf{0}\right\}$
   (iv) The *Column Space* of $\mathbf{A}^T$: $\left\{\tilde{\mathbf{b}} \middle| \mathbf{A}^T\,\mathbf{x} = \tilde{\mathbf{b}} \text{ is solvable}\right\} \equiv$ Span of the rows of $\mathbf{A}$

(9) The Column Space of A is often also called the *Range* of $\mathbf{A}$.

(10) The Null Space of A is often also called the *Kernel* of $\mathbf{A}$.

(11) The Column Space of A^T is commonly also called the *Row Space* of $\mathbf{A}$.

(12) The Null Space of A^T is commonly also called the *Left Null Space* of $\mathbf{A}$

(13) Solutions to the system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ *exist* if and only if $\mathbf{b}$ is in the column space of $\mathbf{A}$

(14) Solutions to the system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ are *unique* if and only there are no **nonzero** vectors in the null space of $\mathbf{A}$.

(15) Gaussian elimination provides all of the information necessary to determine the dimensions of and bases for the four fundamental matrix subspaces, specifically when elementary row operations are used to reduce

$$\mathbf{A} \rightarrow \mathbf{U} \qquad \text{(or} \quad [\mathbf{A} \vdots \mathbf{b}] \rightarrow [\mathbf{U} \vdots \mathbf{z}] \text{)}$$

then:

(i) Both $\mathbf{A}$ and $\mathbf{U}$ have the same row space. Therefore the **non-zero** rows of $\mathbf{U}$ can be used as a basis for the Row($\mathbf{A}$). However, because the rows of $\mathbf{U}$ may have been formed by elementary row operations which included row interchanges, the corresponding rows of $\mathbf{A}$ are not necessarily linearly independent.

(ii) $\mathbf{A}$ and $\mathbf{U}$ generally have **different** column spaces. However, the columns of $\mathbf{A}$ corresponding to the pivot columns of $\mathbf{U}$ are linearly independent, while the columns of $\mathbf{A}$ corresponding to the free variable columns of $\mathbf{U}$ are linearly dependent on them. Therefore the columns of $\mathbf{A}$ corresponding to the pivot columns of $\mathbf{U}$ can be used as a basis for the column space of $\mathbf{A}$.

(iii) The dimension of the null space of $\mathbf{A}$ is the number of free variable columns in $\mathbf{U}$. Finding a basis for the null space requires substituting the appropriate free variables into the corresponding homogeneous system of equation.

(iv) The number of pivots in the echelon matrix $\mathbf{U}$ (i.e. the *rank*) is the dimension of both the row and column spaces of that matrix.

(16) Therefore, if, for a given set of vectors in $\mathbb{R}^n$, one wishes to find:

(i) The dimension of and *any* basis for the subspace they span, simply make them the *rows* of a matrix and reduce that matrix to echelon form,

(ii) The dimension of and a subset of the *original* set of vectors than can be used as a basis for the subspace they span, simply make them the *columns* of a matrix, reduce that matrix to echelon form, and take the columns of the original matrix corresponding to the pivot columns in the echelon one.